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Slug flow heat transfer in circular ducts with viscous dissipation and convective boundary conditions

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Abstract—Forced convection in circular ducts with slug flow and viscous dissipation is analyzed. The temperature field and the local Nusselt number in the thermal entrance region are determined analytically when the tube wall exchanges heat with an external fluid by a uniform convection coefficient. The evaluations are performed for an arbitrary axial distribution of the external-fluid reference temperature. The temperature field and the local Nusselt number for a prescribed axial distribution of wall temperature are obtained by taking the limit of an infinite Biot number. Two cases are analyzed in detail: the reference temperature of the external fluid is uniform; the reference temperature of the external fluid changes linearly in the axial direction. © 1997 Elsevier Science Ltd.

INTRODUCTION

Recently, a model of slug flow forced convection in ducts which takes into account the effect of viscous dissipation has been proposed [1]. This model is based on the assumption that the viscous dissipation of heat occurs in an infinitely thin fluid layer next to the wall. As a consequence, the power generated per unit volume by viscous dissipation is distributed in the duct section as a Dirac's delta centered next to the duct wall. By this model, the thermally developed behavior of the temperature field and of the local Nusselt number has been determined for prescribed axial distributions of the wall heat flux [1]. Then, the temperature field and the local Nusselt number have been determined analytically in the thermal entrance region for an arbitrary axial distribution of the wall heat flux [2]. The solution has been employed to analyze the particular cases of uniform wall heat flux, linearly varying wall heat flux and exponentially varying wall heat flux [2].

Several papers dealing with slug flow forced convection in circular ducts are present in the literature. In most of these papers, slug flow forced convection is considered as an approximation of laminar flow in the hydrodynamic entrance region of fluids with a very low Prandtl number. Moreover, slug flow approximates also the fully developed behavior of the velocity profile for a power-law fluid with a very small power-law index. However, with the only exception of refs. [1–2], all previous investigations of slug flow forced convection neglect the effect of viscous dissipation. For instance, in refs. [3, 4] the temperature field and the local Nusselt number are determined in the thermal entrance region both in the case of uniform wall heat flux and in the case of uniform wall temperature. Singh [5] determines the temperature field in the thermal entrance region for a fluid with a uniform internal heat generation and a uniform wall temperature. The thermally developing temperature field is analyzed also for convective boundary conditions [6–9]. In particular, in ref. [9] the effect of axial heat conduction in the fluid is taken into account. Tyagi and Nigam [7], as well as Javeri [8], find approximate expressions of the temperature field by employing Galerkin–Kantorowich method of variational calculus.

The aim of this paper is to outline the importance of the effect of viscous dissipation for the behavior of the thermally developing temperature field, when boundary conditions of the third kind are prescribed at the duct wall. In order to perform this analysis, the model proposed in refs. [1, 2] is employed. Moreover, the temperature field for boundary conditions of the first kind is obtained in the present paper as a particular case. In fact it is well known that, by taking the limit of an infinite Biot number, the convective boundary condition yields the boundary condition of prescribed wall temperature.

MATHEMATICAL MODEL

In this section, the energy equation and the boundary conditions are written in a dimensionless form and the Laplace transform of the temperature field is

NOMENCLATURE

a	dimensionless slope employed in the case of a linear distribution of external-fluid reference temperature
A_n	dimensionless coefficients employed in the Appendix
Bi	Biot number defined in equation (8)
Br	Brinkman number defined in equation (8)
g	arbitrary function of η employed in the Appendix
h_e	external convection coefficient [W m^{-2} K^{-1}]
i	$=\sqrt{-1}$, imaginary unit
I_v	modified Bessel function of first kind and order v
J_v	Bessel function of first kind and order v
k	thermal conductivity [W m^{-1} K^{-1}]
Nu	Nusselt number defined in equation (31)
n, m	positive integer numbers
p	Laplace transformed variable
Pe	Peclet number defined in equation (8)
r	radial coordinate [m]
r_0	radius of the tube [m]
$Re()$	real part of a complex number
$Res(;)$	residue of a complex function at a pole
T	temperature [K]
T_0	inlet temperature [K]
T_f	reference temperature of the external fluid [K]
u_n	dimensionless function of ξ defined in equation (30)
U_0	uniform axial component of the velocity field [m s^{-1}]
x	axial coordinate [m]
x_0	reference axial position such that $T_f(x_0) \neq T_0$ [m]
X	arbitrary function of η and ξ employed in equation (32)
y	arbitrary real variable.

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
β	$=i\sqrt{p}/2$, dimensionless variable
β_n	nth real positive root of equation (23)
γ	dimensionless function of ξ and Bi defined in equation (37)
δ	Dirac's delta distribution
ζ	$=\sqrt{p}/2$, dimensionless variable
η	dimensionless radial coordinate defined in equation (8)
ϑ	dimensionless temperature defined in equation (8)
ϑ_0	dimensionless function of η and ξ defined by equation (17)
μ	dynamic viscosity coefficient [Pa s]
ξ	dimensionless axial coordinate defined in equation (8)
σ	dimensionless constant employed in equation (24)
φ	dimensionless function of ξ and Bi defined in equation (36)
ϕ_0	power generated by viscous dissipation per unit length of the tube [W m^{-1}]
Φ	viscous dissipation function [s^{-2}]
χ	dimensionless function of ξ and Bi defined in equation (38)
Ψ	dimensionless function of ξ defined in equation (8)
ω	dimensionless function of ξ and Bi defined in equation (39)
Ω	dimensionless function of ξ defined by equation (16).

Superscripts and subscripts

$\sim$	Laplace transformed function
$'$	dummy integration variable
b	bulk value of a function, defined in equation (32).

determined for an arbitrary axial distribution of the external-fluid reference temperature.

Let us consider a slug flow in a circular duct such that the axial heat conduction in the fluid is negligible and both the thermal diffusivity and the thermal conductivity of the fluid are independent of temperature. The axial component of the fluid velocity is uniform within the duct and is vanishing at the duct wall. Since an infinite velocity gradient occurs at the wall, it can be assumed [1, 2] that a heat generation due to viscous dissipation occurs in an infinitely thin fluid layer adjacent to the wall. The power per unit volume generated by viscous dissipation can be described mathematically by a Dirac's delta distribution, namely

$$\mu\Phi(r) = \phi_0\delta(r-r_0) \tag{1}$$

where ϕ_0 is a constant which represents the power dissipated per unit length of the tube and $\delta(r-r_0)$ is such that the integral with respect to r of $2\pi r\delta(r-r_0)$ in the interval $[0, r_0]$ is equal to 1. To account for the smoothness of $T(r, x)$ on the axis of the tube, the condition $\partial T/\partial r = 0$ at $r = 0$ must hold for every value of x. The inlet temperature profile is uniform with value T_0 and the duct wall exchanges heat with an external fluid having a uniform convection coefficient h_e. The reference temperature $T_f(x)$ of the external fluid is axially varying and x_0 is an axial position such that $T_f(x_0) \neq T_0$. Then, the temperature field fulfils the equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{U_0}{\alpha}\frac{\partial T}{\partial x} - \frac{\phi_0}{k}\delta(r-r_0) \tag{2}$$

$$k\frac{\partial T}{\partial r}\bigg|_{r\to r_0} = h_e[T_f(x)-T(r_0,x)], \quad \frac{\partial T}{\partial r}\bigg|_{r\to 0} = 0 \tag{3}$$

$$T(r,0) = T_0. \tag{4}$$

It is easily verified that equations (2)–(4) can be rewritten as [1, 2]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{U_0}{\alpha}\frac{\partial T}{\partial x} \tag{5}$$

$$k\frac{\partial T}{\partial r}\bigg|_{r\to r_0} = h_e[T_f(x)-T(r_0,x)] + \frac{\phi_0}{2\pi r_0}, \quad \frac{\partial T}{\partial r}\bigg|_{r\to 0} = 0 \tag{6}$$

$$T(r,0) = T_0. \tag{7}$$

Let us define the dimensionless quantities

$$\vartheta = \frac{T-T_0}{T_f(x_0)-T_0}, \quad \eta = \frac{r}{r_0}, \quad Pe = \frac{2U_0 r_0}{\alpha}$$

$$\xi = \frac{x}{2r_0\,Pe}, \quad Br = \frac{\phi_0}{2\pi k[T_f(x_0)-T_0]}$$

$$Bi = \frac{h_e r_0}{k}, \quad \Psi(\xi) = \frac{T_f(2r_0\,Pe\,\xi)-T_0}{T_f(x_0)-T_0}. \tag{8}$$

On account of equation (8), equations (5)–(7) can be rewritten as

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \vartheta}{\partial \eta}\right) = \frac{1}{4}\frac{\partial \vartheta}{\partial \xi} \tag{9}$$

$$\frac{\partial \vartheta}{\partial \eta}\bigg|_{\eta\to 1} = -Bi\,\vartheta(1,\xi) + Br + Bi\,\Psi(\xi), \quad \frac{\partial \vartheta}{\partial \eta}\bigg|_{\eta\to 0} = 0 \tag{10}$$

$$\vartheta(\eta,0) = 0. \tag{11}$$

By introducing the Laplace transform of the dimensionless temperature

$$\tilde{\vartheta}(\eta,p) = \int_0^{+\infty} e^{-p\xi}\vartheta(\eta,\xi)\,d\xi \tag{12}$$

and by employing the properties of Laplace transforms [10], equations (9) and (11) yield

$$\eta\frac{\partial^2\tilde{\vartheta}}{\partial \eta^2} + \frac{\partial \tilde{\vartheta}}{\partial \eta} - \eta\frac{p}{4}\tilde{\vartheta} = 0 \tag{13}$$

while equation (10) can be rewritten as

$$\frac{\partial \tilde{\vartheta}}{\partial \eta}\bigg|_{\eta\to 1} = -Bi\,\tilde{\vartheta}(1,p) + \frac{Br + Bi\,\tilde{\Psi}(p)p}{p}, \quad \frac{\partial \tilde{\vartheta}}{\partial \eta}\bigg|_{\eta\to 0} = 0. \tag{14}$$

On account of the properties of Bessel functions [11], the solution of equations (13) and (14) is given by

$$\tilde{\vartheta}(\eta,p) = \frac{(Br+Bi\,\tilde{\Psi}(p)p)I_0(\eta\zeta)}{p[\zeta I_1(\zeta)+Bi\,I_0(\zeta)]} = \tilde{\Omega}(p)\tilde{\vartheta}_0(\eta,p) \tag{15}$$

where $\zeta = \sqrt{p}/2$ and $\tilde{\Omega}(p)$, $\tilde{\vartheta}_0(\eta,p)$ are given by

$$\tilde{\Omega}(p) = \frac{Br+Bi\,\tilde{\Psi}(p)p}{Br+Bi} \tag{16}$$

$$\tilde{\vartheta}_0(\eta,p) = \frac{(Br+Bi)I_0(\eta\zeta)}{p[\zeta I_1(\zeta)+Bi\,I_0(\zeta)]}. \tag{17}$$

Equation (15) and the convolution property of Laplace transforms [10] ensure that

$$\vartheta(\eta,\xi) = \int_0^{\xi}\Omega(\xi')\vartheta_0(\eta,\xi-\xi')\,d\xi' \tag{18}$$

where $\Omega(\xi)$ is the inverse Laplace transform of $\tilde{\Omega}(p)$ and, on account of equation (17) and of the properties of Laplace transforms [10], can be expressed as

$$\Omega(\xi) = \frac{Br+Bi\,\Psi(0)}{Br+Bi}\delta(\xi) + \frac{Bi}{Br+Bi}\frac{d\Psi(\xi)}{d\xi} \tag{19}$$

where $\delta(\xi)$ is such that its integral with respect to ξ in the interval $[0, +\infty]$ is equal to 1. Equations (18) and (19) yield

$$\vartheta(\eta,\xi) = \frac{Br+Bi\,\Psi(0)}{Br+Bi}\vartheta_0(\eta,\xi) + \frac{Bi}{Br+Bi}\int_0^{\xi}\frac{d\Psi(\xi')}{d\xi'}\vartheta_0(\eta,\xi-\xi')\,d\xi'. \tag{20}$$

EVALUATION OF THE TEMPERATURE FIELD

In this section, the inverse Laplace transform of the right-hand side of equation (17) is evaluated. Moreover, the dimensionless temperature field and the local Nusselt number are determined. Finally, the case $Bi \to \infty$, which corresponds to the boundary condition of a prescribed axial distribution of wall temperature, is analyzed.

By defining the variable $\beta = i\zeta = i\sqrt{p}/2$ and by employing the identities [11]

$$I_0(-i\beta) = J_0(\beta), \quad I_1(-i\beta) = i^{-1}J_1(\beta) \tag{21}$$

equation (17) can be rewritten as

$$\tilde{\vartheta}_0(\eta,p) = -\frac{(Br+Bi)J_0(\eta\beta)}{p[\beta J_1(\beta)-Bi\,J_0(\beta)]}. \tag{22}$$

Equation (22) reveals that $\tilde{\vartheta}_0(\eta,p)$ has a simple pole for $p = 0$ and an infinite sequence of simple poles for

$p = -4\beta_n^2$, where $\{\beta_n\}$ is the sequence of real positive roots of the transcendental equation

$$\beta J_1(\beta) - Bi\, J_0(\beta) = 0. \qquad (23)$$

Function $\vartheta_0(\eta, \xi)$ can be evaluated by the inversion formula for Laplace transforms [10]

$$\vartheta_0(\eta,\xi) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{p\xi}\tilde{\vartheta}_0(\eta,p)\,dp \qquad (24)$$

where σ is any real number such that all singularities of the function $\tilde{\vartheta}_0(\eta,p)$ lie in the complex p plane to the left of the line $Re(p) = \sigma$. Since $\tilde{\vartheta}_0(\eta,p)$ has no branch point, the integral on the right-hand side of equation (24) can be evaluated by a contour integration of $e^{p\xi}\tilde{\vartheta}_0(\eta,p)$ on a semicircular closed path which lies to the left of the line $Re(p) = \sigma$ and is centered at $p = \sigma$ [10]. On account of Cauchy's residue theorem, if one lets the radius of the semicircular path tend to infinity, one can rewrite equation (24) as

$$\vartheta_0(\eta, \xi) = \mathrm{Res}(e^{p\xi}\tilde{\vartheta}_0(\eta,p)\,; p = 0) + \sum_{n=1}^{\infty} \mathrm{Res}(e^{p\xi}\tilde{\vartheta}_0(\eta,p)\,; p = -4\beta_n^2). \qquad (25)$$

As a consequence of the properties of Bessel functions [11], the residues which appear in equation (25) can be expressed as

$$\mathrm{Res}(e^{p\xi}\tilde{\vartheta}_0(\eta,p)\,; p = 0) = 1 + \frac{Br}{Bi} \qquad (26)$$

$$\mathrm{Res}(e^{p\xi}\tilde{\vartheta}_0(\eta,p)\,; p = -4\beta_n^2) = -\frac{2\,Bi(Br + Bi)\,e^{-4\beta_n^2\xi}}{\beta_n(Bi^2 + \beta_n^2)}\frac{J_0(\beta_n\eta)}{J_1(\beta_n)}. \qquad (27)$$

Equations (25)–(27) yield

$$\vartheta_0(\eta, \xi) = 1 + \frac{Br}{Bi} - 2\,Bi(Br + Bi) \times \sum_{n=1}^{\infty} \frac{J_0(\beta_n\eta)}{\beta_n(Bi^2 + \beta_n^2)J_1(\beta_n)} e^{-4\beta_n^2\xi}. \qquad (28)$$

On account of equations (20) and (28), the dimensionless temperature field can be written as

$$\vartheta(\eta, \xi) = \Psi(\xi) + \frac{Br}{Bi} - 2\,Bi \times \sum_{n=1}^{\infty} \frac{J_0(\beta_n\eta)\{Br + Bi\,[\Psi(0) + u_n(\xi)]\}}{\beta_n(Bi^2 + \beta_n^2)J_1(\beta_n)} e^{-4\beta_n^2\xi} \qquad (29)$$

where $u_n(\xi)$ is given by

$$u_n(\xi) = \int_0^{\xi} \frac{d\Psi(\xi')}{d\xi'} e^{4\beta_n^2\xi'}\,d\xi'. \qquad (30)$$

As a consequence of equation (3), the local Nusselt number can be expressed as

$$Nu(\xi) = \frac{2r_0}{k}\frac{h_e[T_f(x) - T(r_0,x)]}{T(r_0,x) - T_b(x)} = 2\,Bi\frac{\Psi(\xi) - \vartheta(1,\xi)}{\vartheta(1,\xi) - \vartheta_b(\xi)} \qquad (31)$$

where the bulk value of an arbitrary function $X(\eta, \xi)$ is given by

$$X_b(\xi) = \frac{2}{r_0^2}\int_0^{r_0} X(\eta,\xi)r\,dr = 2\int_0^1 X(\eta,\xi)\eta\,d\eta. \qquad (32)$$

As a consequence of equation (29) and of the identity [11]

$$\int_0^{\beta_n} yJ_0(y)\,dy = \beta_n J_1(\beta_n) \qquad (33)$$

the bulk value of the dimensionless temperature can be written as

$$\vartheta_b(\xi) = \Psi(\xi) + \frac{Br}{Bi} - 4\,Bi \times \sum_{n=1}^{\infty} \frac{Br + Bi\,[\Psi(0) + u_n(\xi)]}{\beta_n^2(Bi^2 + \beta_n^2)} e^{-4\beta_n^2\xi}. \qquad (34)$$

Equations (29), (31) and (34) yield

$$Nu(\xi) = \frac{Br - 2\,Bi\,[Br + Bi\,\Psi(0)]\varphi(\xi, Bi) - 2\,Bi^2\,\gamma(\xi, Bi)}{[Br + Bi\,\Psi(0)]\chi(\xi, Bi) + Bi\,\omega(\xi, Bi)} \qquad (35)$$

where

$$\varphi(\xi, Bi) = \sum_{n=1}^{\infty} \frac{e^{-4\beta_n^2\xi}}{Bi^2 + \beta_n^2} \qquad (36)$$

$$\gamma(\xi, Bi) = \sum_{n=1}^{\infty} \frac{u_n(\xi)\,e^{-4\beta_n^2\xi}}{Bi^2 + \beta_n^2} \qquad (37)$$

$$\chi(\xi, Bi) = \sum_{n=1}^{\infty} \frac{(\beta_n^2 - 2\,Bi)\,e^{-4\beta_n^2\xi}}{\beta_n^2(Bi^2 + \beta_n^2)} \qquad (38)$$

$$\omega(\xi, Bi) = \sum_{n=1}^{\infty} \frac{(\beta_n^2 - 2\,Bi)u_n(\xi)\,e^{-4\beta_n^2\xi}}{\beta_n^2(Bi^2 + \beta_n^2)}. \qquad (39)$$

In Table 1, values of the functions $\varphi(\xi, Bi)$ and $\chi(\xi, Bi)$ are reported for $Bi = 0.1$, $Bi = 1$ and $Bi = 10$. Moreover, in the same table, values of the limits of $Bi^2\,\varphi(\xi, Bi)$ and of $Bi\,\chi(\xi, Bi)$ for $Bi \to \infty$ are reported. The values for $\xi = 0$ reported in Table 1 are obtained by employing the identities proved in the Appendix.

The case of a prescribed axial distribution of wall temperature is easily investigated by taking the limits for $Bi \to \infty$ of the right-hand sides of equations (29), (34) and (35). Indeed, by employing equations (36)–(39), one obtains

Table 1. Values of the functions $\varphi(\xi, Bi)$ and $\chi(\xi, Bi)$ defined by equations (36) and (38)

	$Bi = 0.1$		$Bi = 1$		$Bi = 10$		$Bi \to \infty$	
ξ	$\varphi(\xi, Bi)$	$-\chi(\xi, Bi) \times 10$	$\varphi(\xi, Bi)$	$-\chi(\xi, Bi) \times 10$	$\varphi(\xi, Bi) \times 10$	$-\chi(\xi, Bi) \times 10$	$Bi^2\varphi(\xi, Bi)$	$-Bi\chi(\xi, Bi)$
0.00000	5.0000	0.0000	0.5000	0.0000	0.5000	0.0000	$+\infty$	0.5000
0.00010	4.9886	0.1097	0.4888	0.1079	0.4038	0.0928	13.8533	0.4776
0.00015	4.9861	0.1334	0.4863	0.1308	0.3860	0.1090	11.2647	0.4727
0.00020	4.9839	0.1532	0.4842	0.1498	0.3719	0.1215	9.7215	0.4685
0.00030	4.9802	0.1859	0.4808	0.1808	0.3500	0.1406	7.8909	0.4615
0.00040	4.9771	0.2130	0.4778	0.2062	0.3330	0.1548	6.7994	0.4557
0.00050	4.9744	0.2365	0.4753	0.2281	0.3191	0.1661	6.0545	0.4505
0.00100	4.9635	0.3253	0.4653	0.3090	0.2723	0.2012	4.2056	0.4307
0.00200	4.9479	0.4420	0.4514	0.4108	0.2222	0.2317	2.8970	0.4031
0.00300	4.9356	0.5246	0.4410	0.4794	0.1928	0.2446	2.3166	0.3825
0.00400	4.9252	0.5896	0.4324	0.5311	0.1724	0.2504	1.9701	0.3655
0.00500	4.9159	0.6435	0.4248	0.5721	0.1571	0.2526	1.7332	0.3507
0.01000	4.8782	0.8247	0.3961	0.6958	0.1132	0.2437	1.1430	0.2952
0.02000	4.8214	1.0047	0.3573	0.7827	0.0762	0.2071	0.7195	0.2235
0.03000	4.7746	1.0890	0.3289	0.7933	0.0578	0.1725	0.5255	0.1745
0.04000	4.7324	1.1300	0.3055	0.7739	0.0460	0.1430	0.4040	0.1376
0.05000	4.6929	1.1482	0.2851	0.7414	0.0374	0.1184	0.3168	0.1089
0.10000	4.5102	1.1351	0.2066	0.5533	0.0143	0.0458	0.0989	0.0342
0.15000	4.3375	1.0932	0.1506	0.4041	0.0055	0.0177	0.0311	0.0108
0.20000	4.1715	1.0515	0.1099	0.2948	0.0021	0.0069	0.0098	0.0034
0.25000	4.0119	1.0112	0.0802	0.2150	0.0008	0.0027	0.0031	0.0011
0.30000	3.8584	0.9725	0.0585	0.1569	0.0003	0.0010	0.0010	0.0003
0.35000	3.7107	0.9353	0.0427	0.1144	0.0001	0.0004	0.0003	0.0001
0.40000	3.5687	0.8995	0.0311	0.0835	0.0000	0.0002	0.0001	0.0000
0.45000	3.4322	0.8651	0.0227	0.0609	0.0000	0.0001	0.0000	0.0000
0.50000	3.3008	0.8320	0.0166	0.0444	0.0000	0.0000	0.0000	0.0000

$$\vartheta(\eta, \xi) = \Psi(\xi) - 2 \sum_{n=1}^{\infty} \frac{J_0(\beta_n \eta)[\Psi(0) + u_n(\xi)]}{\beta_n J_1(\beta_n)} e^{-4\beta_n^2 \xi} \tag{40}$$

$$\vartheta_b(\xi) = \Psi(\xi) - 4 \sum_{n=1}^{\infty} \frac{\Psi(0) + u_n(\xi)}{\beta_n^2} e^{-4\beta_n^2 \xi} \tag{41}$$

$$Nu(\xi) = \left(\Psi(0) \sum_{n=1}^{\infty} e^{-4\beta_n^2 \xi} + \sum_{n=1}^{\infty} u_n(\xi) e^{-4\beta_n^2 \xi} - \frac{Br}{2}\right) \times \left(\Psi(0) \sum_{n=1}^{\infty} \frac{e^{-4\beta_n^2 \xi}}{\beta_n^2} + \sum_{n=1}^{\infty} \frac{u_n(\xi) e^{-4\beta_n^2 \xi}}{\beta_n^2}\right)^{-1}. \tag{42}$$

In the limit $Bi \to \infty$, equation (23) ensures that $\{\beta_n\}$ is the sequence of real positive roots of the equation $J_0(\beta) = 0$. Moreover, in this limit, equation (3) ensures that $T(r_0, x) = T_f(x)$, so that $T_f(x)$ represents the prescribed axial distribution of wall temperature. Let us note that equations (40) and (41) reveal that neither the dimensionless temperature nor its bulk value depend on the Brinkman number. Thus, if the axial distribution of wall temperature is prescribed, these quantities are not affected by viscous dissipation. On the other hand, equation (42) ensures that the local Nusselt number depends on the Brinkman number. This circumstance can be explained as follows. Since the axial distribution of wall temperature is prescribed, the temperature field within the duct is uniquely determined and is independent of the heat generation which occurs in an infinitesimal fluid layer next to the wall. On the other hand, the heat generation affects the heat exchanged by the duct wall and hence the local Nusselt number.

UNIFORM AXIAL DISTRIBUTION OF T_f

In this section, the particular case of a uniform axial distribution of the reference temperature of the external fluid is considered, both for a finite Bi and in the limit $Bi \to \infty$.

If $T_f(x)$ is a constant, equation (8) ensures that $\Psi(\xi) = 1$ and equation (30) yields $u_n(\xi) = 0$. Therefore, as a consequence of equations (37) and (39), functions $\gamma(\xi, Bi)$ and $\omega(\xi, Bi)$ vanish.

Equation (20) yields $\vartheta(\eta, \xi) = \vartheta_0(\eta, \xi)$, so that the dimensionless temperature field is expressed by the right-hand side of equation (28). Indeed, equation (28) ensures that the dimensionless temperature field tends to become uniform for $\xi \to +\infty$, with the value $1 + Br/Bi$. Therefore, on account of equations (8) and (28), when $x \to +\infty$ the temperature field becomes uniform, with the value $T_f + \phi_0/(2\pi r_0 h_e)$. This result can be useful for the following reason: if one knows the values of T_f, h_e and r_0, a measurement of the asymptotic value of the fluid temperature yields the value of the power generated by viscous dissipation per unit length of the tube, ϕ_0.

In the case $Br = 0$, the expression of the dimensionless temperature field given by the right-hand side of equation (28) agrees with that found by Golos [6], as it can be easily verified by employing equation (23).

Equation (35) shows that, for $Br = -Bi$, the local Nusselt number is singular at every axial position. Indeed, as a consequence of equation (8), when $Br = -Bi$, the power dissipated per unit wall area $\phi_0/(2\pi r_0)$ equals the power per unit wall area exchanged by external convection, $h_e(T_0 - T_f)$. In this case, the fluid within the duct has no net thermal interaction with the duct wall and its temperature profile is everywhere uniform with value T_0, as can be inferred from equations (8) and (28).

It is easily verified that, for any value of Bi, the lowest non-negative root of equation (23) is such that $\beta_1^2 \leqslant 2\,Bi$. Therefore, if Br is such that $-Bi < Br < 0$, equations (35), (36) and (38) ensure that

$$\lim_{\xi \to +\infty} Nu(\xi) = +\infty. \tag{43}$$

On the other hand, if Br is either positive or less than $-Bi$, the limit is given by

$$\lim_{\xi \to +\infty} Nu(\xi) = -\infty. \tag{44}$$

Moreover, for every nonvanishing value of Bi and for $Br = 0$, the limit has a finite value and can be expressed as

$$\lim_{\xi \to +\infty} Nu(\xi) = \frac{2\,Bi\,\beta_1^2}{2\,Bi - \beta_1^2}. \tag{45}$$

By employing equation (42) with $u_n(\xi) = 0$, one can verify that, in the limit $Bi \to \infty$, the local Nusselt number fulfils equation (43) if Br is negative, while it fulfils equation (44) if Br is positive. If $Br = 0$ and $Bi \to \infty$, the asymptotic value of the Nusselt number is given by

$$\lim_{\xi \to +\infty} Nu(\xi) = \beta_1^2 = 5.7832 \tag{46}$$

where β_1 is the lowest positive root of the equation $J_0(\beta) = 0$. In the case $Br = 0$, equations (40) and (42) with $\Psi(\xi) = 1$ and $u_n(\xi) = 0$ agree with the results obtained by Burmeister [3] and by Shah and Bhatti [4] in the case of uniform wall temperature.

Values of the fully developed Nusselt number for some values of Bi and $Br = 0$, evaluated by equation (35), are reported in Table 2. In the same table, these values are compared with those obtained by Javeri [8] through an approximate analytical method. A fair agreement is observed between the exact values evaluated by equation (35) and the approximate ones presented in ref. [8]. In Fig. 1, a plot of the fully developed Nusselt number vs Bi for $Br = 0$ is reported. This figure and the values reported in Table 2 illustrate that, in the absence of viscous dissipation, the convective boundary condition yields fully developed values of the Nusselt number which lie between 5.7832 and 8. The former value corresponds to uniform wall temperature ($Bi \to \infty$), while the latter value corresponds to the case $Bi \to 0$ and coincides with that obtained for uniform wall heat flux [1, 3, 4].

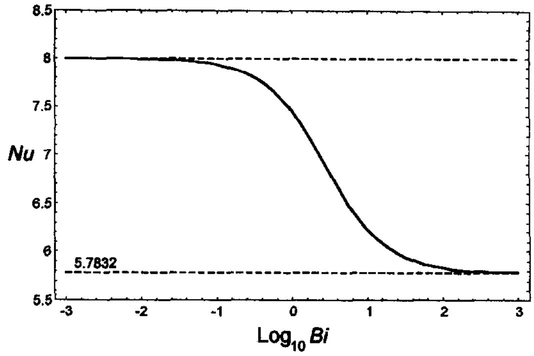

Fig. 1. The fully developed Nusselt number as a function of Bi, for $Br = 0$.

The behavior of the local Nusselt number in the thermal entrance region is illustrated in Figs. 2–5. In Figs. 2 and 3, the case $Bi = 1$ is considered: Fig. 2 refers to $Br > -1$, while Fig. 3 refers to $Br < -1$. As it has been pointed out above, if $Bi = 1$, the value $Br = -1$ yields an infinite local Nusselt number at every axial position. For $Br > 0$, the local Nusselt number decreases for increasing values of ξ and tends to $-\infty$ for $\xi \to +\infty$. For $-1 < Br < 0$, the local Nusselt number decreases for increasing values of ξ until a minimum is reached, then Nu increases and tends to $+\infty$ for $\xi \to +\infty$. The value of ξ which yields the position of the minimum value of Nu is an increasing function of Br. For $Br < -1$, as is shown in Fig. 3, Nu initially increases with ξ, reaches a maximum, then decreases and tends to $-\infty$ for

Table 2. Fully developed values of Nu for various values of Bi and $Br = 0$, compared with those obtained by Javeri [8]

Bi	Nu present work	Nu Javeri [8]	Bi	Nu present work	Nu Javeri [8]
0.000	8.0000	—	5	6.5545	6.547
0.001	7.9993	—	10	6.2299	6.224
0.010	7.9933	—	20	6.0235	—
0.050	7.9670	—	50	5.8835	—
0.100	7.9347	—	100	5.8340	5.832
0.500	7.6994	—	500	5.7935	—
1.000	7.4561	—	1000	5.7883	—
2.000	7.0975	7.087	∞	5.7832	5.783

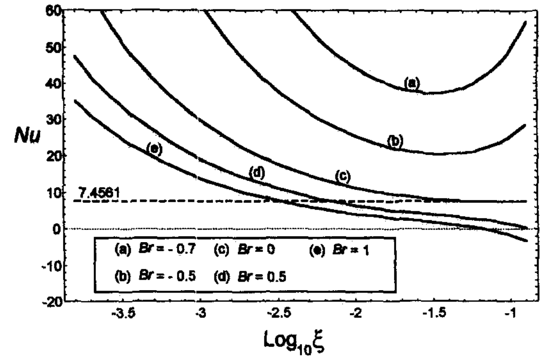

Fig. 2. Behavior of Nu as a function of ξ in the thermal entrance region, in the case of a uniform reference temperature of the external fluid, $Bi = 1$ and $Br > -1$. The dashed line corresponds to the fully developed value of Nu for $Br = 0$.

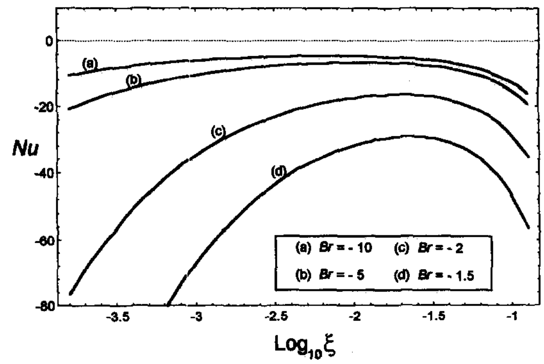

Fig. 3. Behavior of Nu as a function of ξ in the thermal entrance region, in the case of a uniform reference temperature of the external fluid, $Bi = 1$ and $Br < -1$.

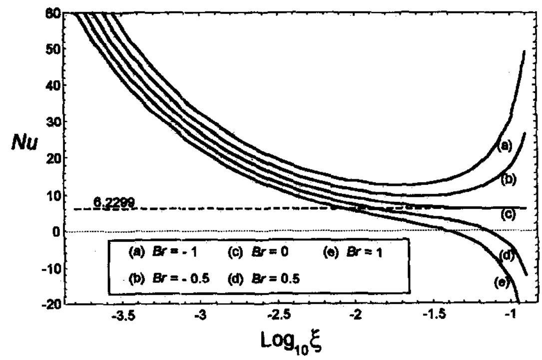

Fig. 4. Behavior of Nu as a function of ξ in the thermal entrance region, in the case of a uniform reference temperature of the external fluid and $Bi = 10$. The dashed line corresponds to the fully developed value of Nu for $Br = 0$.

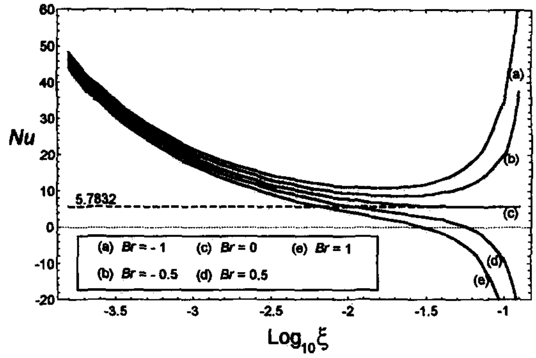

Fig. 5. Behavior of Nu as a function of ξ in the thermal entrance region, in the case of a uniform reference temperature of the external fluid and $Bi \to \infty$. The dashed line corresponds to the fully developed value of Nu for $Br = 0$.

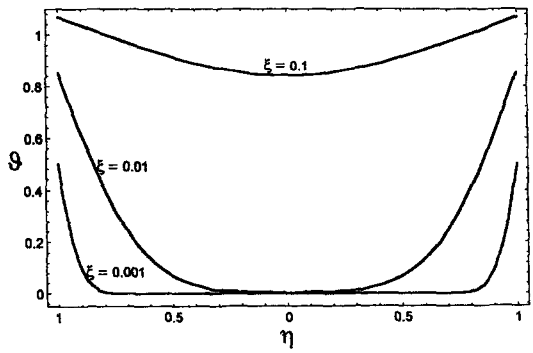

Fig. 6. Behavior of ϑ as a function of η for various values of ξ, in the case of a uniform reference temperature of the external fluid, $Bi = 10$ and $Br = 1$.

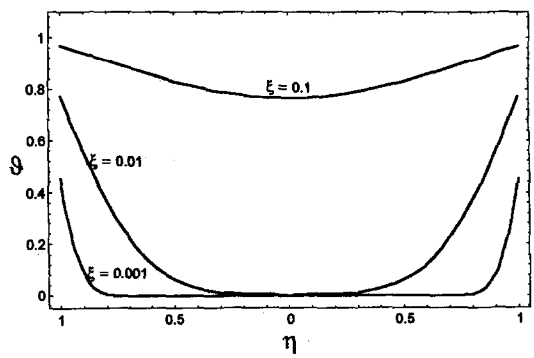

Fig. 7. Behavior of ϑ as a function of η for various values of ξ, in the case of a uniform reference temperature of the external fluid, $Bi = 10$ and $Br = 0$.

$\xi \to +\infty$. Figure 4 refers to $Bi = 10$, while Fig. 5 refers to $Bi \to \infty$, i.e. to uniform wall temperature. Both the curves reported in Fig. 4 and those reported in Fig. 5 display a behavior similar to that of the curves represented in Fig. 2. However, for a given value of the Brinkman number, a comparison between Figs. 2, 4 and 5 shows that the local Nusselt number tends to be smaller for higher values of Bi.

Figures 6–8 refer to $Bi = 10$ and represent the evolution of the dimensionless temperature profile for increasing values of ξ. In Figs. 6–8, the values $Br = 1$, $Br = 0$ and $Br = -1$ are considered, respectively. These figures reveal that, for $\xi = 0.1$, the dimen-

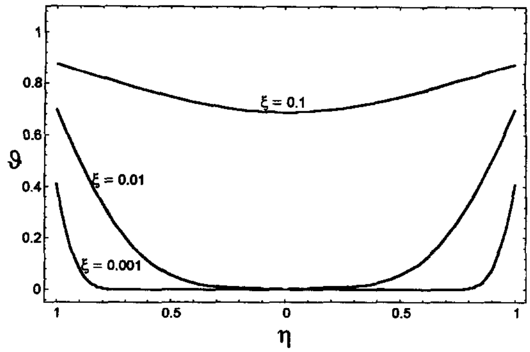

Fig. 8. Behavior of ϑ as a function of η for various values of ξ, in the case of a uniform reference temperature of the external fluid, $Bi = 10$ and $Br = -1$.

sionless temperature profile has almost reached its fully developed form, which is uniform with the value $1 + Br/Bi$.

LINEAR AXIAL DISTRIBUTION OF T_f

In this section, the particular case of a linearly varying reference temperature for the external fluid is considered, both for a finite Bi and in the limit $Bi \to \infty$.

Let us assume that $\Psi(\xi) = 1 + a\xi$, so that equation (30) yields

$$u_n(\xi) = a\frac{e^{4\beta_n^2\xi} - 1}{4\beta_n^2}. \quad (47)$$

On account of equations (23), (29), (47) and of the identities proved in the Appendix, one obtains

$$\vartheta(\eta, \xi) = 1 + \frac{Br}{Bi} + a\left[\xi - \frac{Bi(1-\eta^2)+2}{16\,Bi}\right] - \frac{Bi}{2}\sum_{n=1}^{\infty}\frac{J_0(\beta_n\eta)[4\beta_n^2(Br+Bi) - a\,Bi]}{\beta_n^3(Bi^2+\beta_n^2)J_1(\beta_n)}e^{-4\beta_n^2\xi}. \quad (48)$$

Equation (48) shows that, for high values of ξ, the difference $\vartheta(1,\xi) - \vartheta(\eta,\xi)$ can be approximated as

$$\vartheta(1,\xi) - \vartheta(\eta,\xi) \approx \frac{a}{16}(1-\eta^2). \quad (49)$$

Therefore, for high values of ξ, the difference $\vartheta(1,\xi) - \vartheta(\eta,\xi)$ depends neither on Bi nor on Br. Moreover, as a consequence of equations (37), (39), (47) and of the identities proved in the Appendix, functions $\gamma(\xi, Bi)$ and $\omega(\xi, Bi)$ can be expressed as

$$\gamma(\xi, Bi) = \frac{a}{4}\sum_{n=1}^{\infty}\frac{1}{\beta_n^2(Bi^2+\beta_n^2)} - \frac{a}{4}\sum_{n=1}^{\infty}\frac{e^{-4\beta_n^2\xi}}{\beta_n^2(Bi^2+\beta_n^2)} = \frac{a}{16\,Bi^2} - \frac{a}{4}\sum_{n=1}^{\infty}\frac{e^{-4\beta_n^2\xi}}{\beta_n^2(Bi^2+\beta_n^2)} \quad (50)$$

$$\omega(\xi, Bi) = \frac{a}{4}\sum_{n=1}^{\infty}\frac{\beta_n^2 - 2\,Bi}{\beta_n^4(Bi^2+\beta_n^2)} - \frac{a}{4}\sum_{n=1}^{\infty}\frac{(\beta_n^2 - 2\,Bi)\,e^{-4\beta_n^2\xi}}{\beta_n^4(Bi^2+\beta_n^2)} = -\frac{a}{64\,Bi} - \frac{a}{4}\sum_{n=1}^{\infty}\frac{(\beta_n^2 - 2\,Bi)\,e^{-4\beta_n^2\xi}}{\beta_n^4(Bi^2+\beta_n^2)}. \quad (51)$$

The local Nusselt number can be evaluated by employing equations (35), (36), (38), (50) and (51). It is easily verified that

$$\lim_{\xi\to+\infty} Nu(\xi) = 8 - \frac{64\,Br}{a}. \quad (52)$$

Equation (52) reveals that the asymptotic value of the Nusselt number is independent of Bi and, if $Br = 0$, is equal to 8 independently of a. Therefore, equation (52) holds also in the particular case of a prescribed linear distribution of wall temperature, i.e. in the limit $Bi \to \infty$. In this limit, the local Nusselt number can be evaluated by employing equations (42), (47) and the identities proved in the Appendix: one obtains

$$Nu(\xi) = \left(\sum_{n=1}^{\infty}e^{-4\beta_n^2\xi} - \frac{a}{4}\sum_{n=1}^{\infty}\frac{e^{-4\beta_n^2\xi}}{\beta_n^2} + \frac{a}{16} - \frac{Br}{2}\right) \times\left(\sum_{n=1}^{\infty}\frac{e^{-4\beta_n^2\xi}}{\beta_n^2} - \frac{a}{4}\sum_{n=1}^{\infty}\frac{e^{-4\beta_n^2\xi}}{\beta_n^4} + \frac{a}{128}\right)^{-1}. \quad (53)$$

As a consequence of equation (48) and of equations (35), (50), (51), it is easily verified that the limits for $a \to 0$ of $\vartheta(\eta,\xi)$ and of $Nu(\xi)$ obtained for a linear distribution of T_f yield $\vartheta(\eta,\xi)$ and $Nu(\xi)$ for a uniform distribution of T_f. However, the limits for $\xi \to +\infty$ and for $a \to 0$ of $Nu(\xi)$ cannot be interchanged. In fact, if one takes the limit $a \to 0$ of the right-hand side of equation (52), one does not recover the correct asymptotic behavior of $Nu(\xi)$ for a uniform distribution of T_f, which has been described in the previous section. For instance, if $Br = 0$, the asymptotic value of $Nu(\xi)$ for a uniform distribution of T_f depends on Bi and is given by equation (45), while the limit for $a \to 0$ of the right-hand side of equation (52) is equal to 8 for every value of Bi. This circumstance, is similar to that described by Grigull and Tratz [12] with reference to Pouiseuille-flow forced convection in a circular duct with a linear axial distribution of wall temperature.

The behavior of the local Nusselt number in the thermal entrance region is reported in Fig. 9 for $Bi = 1$, $a = 1$ and for the values of the Brinkman number $Br = 0.1$, $Br = 0.05$, $Br = 0$, $Br = -0.05$, $Br = -0.1$. Moreover, the corresponding asymptotic values of $Nu(\xi)$ are reported in this figure. For non-positive values of Br, the plots reported in Fig. 9 show that $Nu(\xi)$ reaches a minimum and then increases towards its asymptotic value. On the other hand, if Br is positive, $Nu(\xi)$ decreases and tends to its asymptotic value. Indeed, this figure shows that the effect of vis-

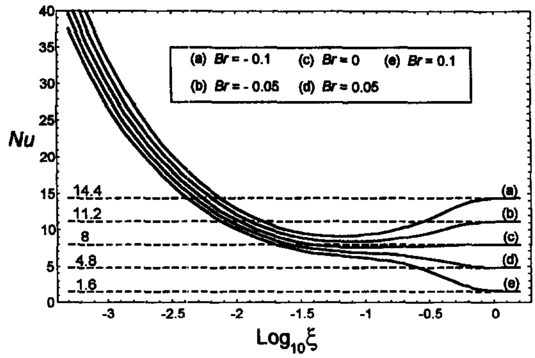

Fig. 9. Behavior of Nu as a function of ξ in the thermal entrance region, in the case of a linearly varying reference temperature of the external fluid, $Bi = 1$ and $a = 1$. The dashed lines correspond to the fully developed values of Nu.

cous dissipation is particularly relevant in the fully developed region, i.e. for $\xi \gtrsim 1$.

CONCLUSIONS

Forced convection with slug flow in a circular duct has been investigated in the case of convective boundary conditions, by taking into account the effect of viscous dissipation in the fluid. An analytical solution of the stationary energy equation has been obtained for an axially varying reference temperature of the external fluid. The case of a prescribed axial distribution of wall temperature has been studied by taking the limit $Bi \to \infty$ of the general solution. In particular, two axial distributions of the external fluid reference temperature T_f have been considered: a uniform distribution, a linear distribution. In the case of a uniform distribution, it has been shown that, for any value of Bi and for any non-vanishing value of Br, the local Nusselt number does not converge to an asymptotic value in the limit $\xi \to +\infty$. On the contrary, for any linear distribution of T_f with a non-vanishing slope, a fully developed value of $Nu(\xi)$ exists for any value of Bi and of Br. It has been pointed out that this fully developed value does not depend on the Biot number.

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APPENDIX

An arbitrary function $g(\eta)$ which is integrable in the interval $0 \leqslant \eta \leqslant 1$ can be expanded as follows [13]:

$$g(\eta) = \sum_{n=1}^{\infty} A_n J_0(\beta_n \eta) \quad \text{(A1)}$$

where β_n are the positive roots of equation (23) and A_n are given by

$$A_n = \frac{2}{J_0(\beta_n)^2 + J_1(\beta_n)^2} \int_0^1 \eta g(\eta) J_0(\beta_n \eta)\,d\eta$$
$$= \frac{2}{J_0(\beta_n)^2(\beta_n^2 + Bi^2)} \int_0^{\beta_n} y g\left(\frac{y}{\beta_n}\right) J_0(y)\,dy. \quad \text{(A2)}$$

For an arbitrary integer m the following identity holds [13]:

$$\int y^m J_0(y)\,dy = y^m J_1(y) + (m-1) y^{m-1} J_0(y)$$
$$-(m-1)^2 \int y^{m-2} J_0(y)\,dy. \quad \text{(A3)}$$

By expanding the functions 1, η^2, η^4, equations (A1)–(A3) yield

$$1 = 2\,Bi \sum_{n=1}^{\infty} \frac{J_0(\beta_n \eta)}{J_0(\beta_n)(\beta_n^2 + Bi^2)} \quad \text{(A4)}$$

$$\eta^2 = 1 + \frac{2}{Bi} - 8\,Bi \sum_{n=1}^{\infty} \frac{J_0(\beta_n \eta)}{J_0(\beta_n)(\beta_n^2 + Bi^2)\beta_n^2} \quad \text{(A5)}$$

$$\eta^4 = 4\left(1 + \frac{2}{Bi}\right)\eta^2 - \frac{16}{Bi^2} - \frac{12}{Bi} - 3$$
$$+ 128\,Bi \sum_{n=1}^{\infty} \frac{J_0(\beta_n \eta)}{J_0(\beta_n)(\beta_n^2 + Bi^2)\beta_n^4} \quad \text{(A6)}$$

which hold for any value of η and of Bi. In particular, by considering identities (A4)–(A6) for $\eta = 1$, one obtains

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^2 + Bi^2} = \frac{1}{2\,Bi} \qquad \text{(A7)}$$

$$\sum_{n=1}^{\infty} \frac{1}{(\beta_n^2 + Bi^2)\beta_n^2} = \frac{1}{4\,Bi^2} \qquad \text{(A8)}$$

$$\sum_{n=1}^{\infty} \frac{1}{(\beta_n^2 + Bi^2)\beta_n^4} = \frac{1}{32\,Bi^2} + \frac{1}{8\,Bi^3}. \qquad \text{(A9)}$$

In the limit $Bi \to \infty$, β_n represents the nth positive root of the equation $J_0(\beta) = 0$, and equations (A8) and (A9) yield

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^2} = \frac{1}{4}, \quad \sum_{n=1}^{\infty} \frac{1}{\beta_n^4} = \frac{1}{32}. \qquad \text{(A10)}$$